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# On the Laplace transform of a matrix of fundamental solutions for thermoelastic plates

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**Abstract.** The Laplace transform $\hat{D}(x, p)$ of a matrix $D(x, t)$ of fundamental solutions for the partial differential operator describing the time-dependent bending of thermoelastic plates with transverse shear deformation is constructed, and its asymptotic behavior near the origin is investigated. The differential system is reduced to an algebraic one through the application of the Laplace and then Fourier transformations, and all possible cases of roots of the determinant of the latter system are considered. It is shown that in every case, the asymptotic expansion of $\hat{D}(x, p)$ near the origin has the same dominant term. This is an important step in the construction of boundary-element methods for the above time-dependent model because it determines the nature of the singularity of the kernel of the boundary-integral-equations associated with various initial-boundary-value problems for the governing system.

**Key words:** asymptotic behavior, fundamental solutions, thermoelastic plates

## 1. Introduction

Elastic plates play an important role in mechanical structures since they can support loads far in excess of their own weight. In addition, due to their geometric characteristics, thin plates can be studied mathematically by means of two-dimensional models instead of the full and much more complex equations of three-dimensional elasticity.

The first plate-bending model was proposed by Kirchhoff in 1850 [1]. Making a number of simplifying hypotheses, he arrived at the conclusion that, in terms of Cartesian coordinates $(x_1, x_2, x_3)$ with $(x_1, x_2)$ in the middle plane of the plate, the displacement field should be of the form $(x_3u_1, x_3u_2, u_3)$, where the functions $u_i = u_i(x_1, x_2), i = 1, 2, 3,$ satisfy

$$u_1 = -u_{3,1}, \quad u_2 = -u_{3,2}, \quad \Delta\Delta u_3 = \frac{q}{D},$$

$q$ being the total load, $D$ the modulus of rigidity of the material, and $\Delta$ the two-dimensional Laplacian. It is clear that the nonhomogeneous biharmonic equation for $u_3$ cannot take more than two boundary conditions, which means that the twisting moment and the two components of the bending moment cannot be prescribed independently on the boundary. This drawback is compounded by the fact that, in Kirchhoff's theory, the transverse shear force is identically zero throughout the plate.

Since there are numerous cases where the transverse shear force is not negligible and each of the three moments must be given on the contour, the need arose for more refined models with a more sophisticated mathematical content. One such model was proposed by Reissner in 1944 [2–4], who started with the stress tensor, postulating a linear dependence on $x_3$ for the components $t_{\alpha\beta}$, $\alpha, \beta \in \{1,2\}$, and a certain type of parabolic dependence on $x_3$ for the components $t_{\alpha 3}$, $\alpha = 1,2$. This model accepts three independent boundary conditions, but does not yield the explicit expressions of the displacements. Another model, proposed by Mindlin in 1951 [5], is based on Kirchhoff's kinematic assumption on the displacement field but has no tie between its first two components and the third one, as above; however, it makes use of a correction factor in the constitutive equations, which interferes with its mathematical rigor. Again, this model accepts three boundary conditions. A theory similar to Mindlin's but without any "fudge" factor was studied in [6]. It should be pointed out that all Reissner- and Mindlin-type models also account for the transverse shear force in the plate.

If temperature plays a significant role in the process of bending – in other words, if the plate is regarded as thermoelastic – additional terms and an additional equation must be added to the governing system [7]. The deformation of thermoelastic plates is of interest in a wide variety of practical problems, from microchip production to aerospace industry.

Once a mathematical model has been set up for a physical process or phenomenon, it needs to be properly investigated and shown to be well-posed before it can be used to produce meaningful numerical approximations. There are many techniques for studying the question of existence, uniqueness, and continuous dependence of solutions on the data. One of them, particularly well suited for linear elliptic second-order systems of partial differential equations, is the boundary-integral-equation method, in which the solution is actually constructed explicitly as a layer potential whose density satisfies an integral equation on the boundary of the domain. The kernel of the potential operator is expressed in terms of a matrix of fundamental solutions for the system. If the system is time-dependent, an application of the Laplace transformation usually converts it into an elliptic one, and the kernel of the corresponding boundary-integral equation in the transform domain is the Laplace transform

$$\hat{D}(x,p) = \int_0^\infty e^{-pt} D(x,t)\,dt, \quad x \in \mathbb{R}^2 \setminus \{0\}, \quad \mathfrak{Re}\,p > 0, \tag{1}$$

of the matrix $D(x,t)$ of fundamental solutions for the original system. It is this transformed matrix for the model of bending motion of thermoelastic plates with transverse shear deformation introduced in [7] that we propose to derive and discuss below. Specifically, we are interested in the asymptotic behavior of $\hat{D}(x,p)$ as $|x| \to 0$. Our interest is twofold: on the one hand, this gives us an indication of the nature of the singularity of the kernel of the boundary-integral operator associated with the problem; and on the other hand, as shown in [8], we can later successfully construct numerical methods for this type of linear evolution problem, which are based solely on estimates for the Laplace transform of the fundamental solution and do not require explicit knowledge of the time-dependent fundamental solution itself.

The corresponding matrices in the adiabatic static and dynamic cases were obtained in [6, Section 2.2] and [9], respectively.

## 2. Preliminaries

Consider a homogeneous and isotropic elastic plate of constant thickness $h_0$ which occupies a region $\bar{S} \times [-h_0/2, h_0/2]$ in $\mathbb{R}^3$. The displacement vector at a point $x' \in \bar{S} \times [-h_0/2, h_0/2]$ in this region at $t \geq 0$ is written as

$$v(x',t) = (v_1(x',t), v_2(x',t), v_3(x',t))^{\mathrm{T}},$$

where the superscript T means matrix transposition. The plate temperature measured from a reference absolute temperature $T_0$ is $v_4(x', t)$.

In what follows, we use various symbols to denote physical constants characterizing the plate material. Thus, $\lambda$ and $\mu$ are the Lamé coefficients, satisfying $\lambda+\mu>0$ and $\mu>0$, $\rho$ is the density, $c_0$ is the specific heat, $\alpha$ and $k_0$ are the coefficients of thermal expansion and conductivity, and

$$\gamma=\alpha(3\lambda+2\mu), \quad \eta=\frac{\gamma T_0}{k_0}, \quad \varkappa=\frac{k_0}{\rho c_0}.$$

Let $x'=(x, x_3)$, $x=(x_1, x_2)\in\mathbb{R}^2$. In the model of thermoelastic plates with transverse shear deformation proposed in [7] it is assumed that

$$v(x', t)=(x_3u_1(x, t), x_3u_2(x, t), u_3(x, t))^{\mathrm{T}}, \quad v_4(x', t)=x_3\theta(x, t).$$

Then $U(x, t)=\big(u_1(x, t), u_2(x, t), u_3(x, t), \theta(x, t)\big)^{\mathrm{T}}$ satisfies the equation

$$\mathcal{B}_0(\partial_t^2U)(x, t)+\mathcal{B}_1(\partial_tU)(x, t)+(\mathcal{A}U)(x, t)=\mathcal{Q}(x, t), \quad x\in S, \ t>0, \tag{2}$$

where $\mathcal{B}_0=\operatorname{diag}\{\rho h^2, \rho h^2, \rho, 0\}$, $h^2=h_0^2/12$, $\partial_t=\partial/\partial t$, $\mathcal{Q}(x, t)=(q(x, t)^{\mathrm{T}}, q_4(x, t))^{\mathrm{T}}$, $q(x, t)=(q_1(x, t), q_2(x, t), q_3(x, t))^{\mathrm{T}}$ is a combination of the forces and moments acting on the plate and its faces, $q_4(x, t)$ is a combination of the averaged heat-source density and the temperature and heat flux on the faces,

$$\mathcal{B}_1=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \eta\partial_1 & \eta\partial_2 & 0 & \varkappa^{-1}\end{pmatrix}, \quad \mathcal{A}=\begin{pmatrix} & & & h^2\gamma\partial_1\\ & A & & h^2\gamma\partial_2\\ & & & 0\\ 0 & 0 & 0 & -\Delta\end{pmatrix},$$
$$A=\begin{pmatrix} -h^2\mu\Delta-h^2(\lambda+\mu)\partial_1^2+\mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1\\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta-h^2(\lambda+\mu)\partial_2^2+\mu & \mu\partial_2\\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta\end{pmatrix}, \tag{3}$$

and $\partial_\alpha=\partial/\partial x_\alpha$, $\alpha=1, 2$.

Let $D(x, t)$ be a matrix of fundamental solutions for the partial differential operator acting on the left-hand side in (2); that is, $D(x, t)$ is a $(4\times 4)$-matrix such that

$$\begin{aligned}&\mathcal{B}_0(\partial_t^2D)(x, t)+\mathcal{B}_1(\partial_tD)(x, t)+(\mathcal{A}D)(x, t)=\delta(x, t)I, \quad (x, t)\in\mathbb{R}^3,\\ &D(x, t)=0, \quad t\le 0,\end{aligned}$$

where $I$ is the $(4\times 4)$-identity matrix. We note that the Laplace transform $\hat{D}(x, p)$ of $D(x, t)$, defined by (1), being a holomorphic matrix-valued function in the complex half-plane $\mathfrak{Re}\,p>0$ and having at most polynomial growth at infinity, satisfies the transformed equation

$$p^2\mathcal{B}_0\hat{D}(x, p)+p\mathcal{B}_1\hat{D}(x, p)+(\mathcal{A}\hat{D})(x, p)=\delta(x)I, \quad x\in\mathbb{R}^2, \ \mathfrak{Re}\,p>0.$$

Furthermore, its Fourier transform

$$\tilde{D}(\xi, p)=\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}(x,\xi)}\hat{D}(x, p)\,\mathrm{d}x$$

is a solution of the equation

$$p^2\mathcal{B}_0\tilde{D}(\xi, p)+p\mathcal{B}_1(\xi)\tilde{D}(\xi, p)+\mathcal{A}(\xi)\tilde{D}(\xi, p)=I, \quad \xi\in\mathbb{R}^2, \ \mathfrak{Re}\,p>0, \tag{4}$$

where

$$\mathcal{B}_1(\xi)=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\mathrm{i}\eta\xi_1 & -\mathrm{i}\eta\xi_2 & 0 & \varkappa^{-1}\end{pmatrix},\quad \mathcal{A}(\xi)=\begin{pmatrix} & & & -\mathrm{i}h^2\gamma\xi_1\\ & A(\xi) & & -\mathrm{i}h^2\gamma\xi_2\\ & & & 0\\ 0 & 0 & 0 & |\xi|^2\end{pmatrix},$$

and

$$A(\xi)=\begin{pmatrix} h^2\mu|\xi|^2+h^2(\lambda+\mu)\xi_1^2+\mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mathrm{i}\mu\xi_1\\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu|\xi|^2+h^2(\lambda+\mu)\xi_2^2+\mu & -\mathrm{i}\mu\xi_2\\ \mathrm{i}\mu\xi_1 & \mathrm{i}\mu\xi_2 & \mu|\xi|^2\end{pmatrix}.$$

First, we construct the matrix $\tilde{D}(\xi,p)$; then, making use of the inverse Fourier transformation, we find $\hat{D}(x,p)$ from $\tilde{D}(\xi,p)$; finally, we study the asymptotic behavior of $\hat{D}(x,p)$ as $|x|\to 0$.

## 3. The Fourier–Laplace transform of the matrix of fundamental solutions

We write (4) in the form

$$\Theta(\xi,p)\tilde{D}(\xi,p)=I,$$

where

$$\Theta(\xi,p)=p^2\mathcal{B}_0+p\mathcal{B}_1(\xi)+\mathcal{A}(\xi).$$

A straightforward calculation shows that

$$\begin{aligned}\det\Theta(\xi,p)=(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\big\{&(\varkappa^{-1}p+|\xi|^2)\times\\ &\times\big[(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu+h^2(\lambda+\mu)|\xi|^2)-\mu^2|\xi|^2\big]\\ &+\eta h^2\gamma p|\xi|^2(\rho p^2+\mu|\xi|^2)\big\}.\end{aligned}$$

Let $\tilde{\Psi}(\xi,p)=\big[\det\Theta(\xi,p)\big]^{-1}$. Then the entries of the inverse $\Theta^{-1}(\xi,p)=\tilde{D}(\xi,p)$ are

$$\begin{aligned}\tilde{D}_{11}(\xi,p)=\big\{&(\varkappa^{-1}p+|\xi|^2)\big[(\rho p^2+\mu|\xi|^2)\big(\rho h^2p^2+h^2\mu|\xi|^2+\mu+h^2(\lambda+\mu)\xi_2^2\big)-\mu^2\xi_2^2\big]+\\ &+\eta h^2\gamma p\xi_2^2(\rho p^2+\mu|\xi|^2)\big\}\tilde{\Psi}(\xi,p),\\ \tilde{D}_{22}(\xi,p)=\big\{&(\varkappa^{-1}p+|\xi|^2)\big[(\rho p^2+\mu|\xi|^2)\big(\rho h^2p^2+h^2\mu|\xi|^2+\mu+h^2(\lambda+\mu)\xi_1^2\big)-\mu^2\xi_1^2\big]+\\ &+\eta h^2\gamma p\,\xi_1^2(\rho p^2+\mu|\xi|^2)\big\}\tilde{\Psi}(\xi,p),\\ \tilde{D}_{33}(\xi,p)=\big\{&(\varkappa^{-1}p+|\xi|^2)\big[\big(\rho h^2p^2+h^2\mu|\xi|^2+\mu+h^2(\lambda+\mu)\xi_1^2\big)\times\\ &\times\big(\rho h^2p^2+h^2\mu|\xi|^2+\mu+h^2(\lambda+\mu)\xi_2^2\big)-h^4(\lambda+\mu)^2\xi_1^2\xi_2^2\big]+\\ &+\eta h^2\gamma p\,|\xi|^2(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\big\}\tilde{\Psi}(\xi,p),\\ \tilde{D}_{44}(\xi,p)=(&\rho h^2p^2+h^2\mu|\xi|^2+\mu)\big[(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)+\\ &+h^2(\lambda+\mu)|\xi|^2(\rho p^2+\mu|\xi|^2)-\mu^2|\xi|^2\big]\tilde{\Psi}(\xi,p),\\ \tilde{D}_{12}(\xi,p)=\tilde{D}_{21}(\xi,p)=&-\xi_1\xi_2\big\{(\varkappa^{-1}p+|\xi|^2)\big[h^2(\lambda+\mu)(\rho p^2+\mu|\xi|^2)-\mu^2\big]+\\ &+\eta h^2\gamma p(\rho p^2+\mu|\xi|^2)\big\}\tilde{\Psi}(\xi,p),\end{aligned}$$

$$\tilde{D}_{13}(\xi,p)=-\tilde{D}_{31}(\xi,p)=\mathrm{i}\mu\xi_1(\varkappa^{-1}p+|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,t),$$
$$\tilde{D}_{14}(\xi,p)=\mathrm{i}h^2\gamma\xi_1(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{41}(\xi,p)=\mathrm{i}p\eta\xi_1(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{23}(\xi,p)=-\tilde{D}_{32}(\xi,p)=\mathrm{i}\mu\xi_2(\varkappa^{-1}p+|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{24}(\xi,p)=\mathrm{i}h^2\gamma\xi_2(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{42}(\xi,p)=\mathrm{i}p\eta\xi_2(\rho p^2+\mu|\xi|^2)(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{34}(\xi,p)=h^2\gamma\mu|\xi|^2(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p),$$
$$\tilde{D}_{43}(\xi,p)=-\mu\eta p|\xi|^2(\rho h^2p^2+h^2\mu|\xi|^2+\mu)\tilde{\Psi}(\xi,p).$$

To obtain the inverse Fourier transform $\hat{D}(x,p)$ of $\tilde{D}(\xi,p)$, we need to study some properties of the roots of the equation $\det\Theta(\xi,p)=0$. We set $|\xi|^2=s$ in $\det\Theta(\xi,p)$ and regard the polynomial

$$P(s,p)=(\rho h^2p^2+h^2\mu s+\mu)\big\{(\varkappa^{-1}p+s)\big[(\rho p^2+\mu s)\big(\rho h^2p^2+h^2(\lambda+2\mu)s+\mu\big)-\mu^2s\big]+ \\ +\eta h^2\gamma ps(\rho p^2+\mu s)\big\}$$

as a function of the complex variable $s$ for values of the parameter $p\in\mathbb{C}$ such that $\mathfrak{Re}\,p>0$. We denote the zeros of $P(s,p)$ by

$$s_j=-\chi_j^2,\quad \mathfrak{Re}\,\chi_j\geq 0,\quad j=1,2,3,4.$$

In the Appendix A, it is shown that, in fact,

$$\mathfrak{Re}\,\chi_j>0,\quad j=1,2,3,4.$$

## 4. The Laplace transform of the matrix of fundamental solutions

To obtain the Laplace transform $\hat{D}(x,p)$ of the matrix of fundamental solutions, we require the inverse Fourier transform of $\tilde{D}(\xi,p)$; that is,

$$\hat{D}(x,p)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x,\xi)}\tilde{D}(\xi,p)\,\mathrm{d}\xi.$$

It is clear from the expressions of the entries $\tilde{D}_{ij}(\xi,p)$, $i,j=1,2,3,4$, of $\tilde{D}(\xi,p)$ that it suffices to compute

$$\hat{\Psi}(x,p)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x,\xi)}\tilde{\Psi}(\xi,p)\,\mathrm{d}\xi, \tag{5}$$

where, as defined earlier,

$$\tilde{\Psi}(\xi,p)=\big[\det(\Theta(\xi,p)\big]^{-1},\quad \Theta(\xi,p)=p^2\mathcal{B}_0+p\mathcal{B}_1(\xi)+\mathcal{A}(\xi).$$

Since $\tilde{\Psi}(\xi,p)$ depends only on $|\xi|^2$, it is convenient to express (5) in terms of polar coordinates, and write

$$\hat{\Psi}(x,p)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\frac{\rho\mathrm{e}^{-\mathrm{i}|x|\rho\cos\varphi}}{P(\rho^2,p)}\,\mathrm{d}\varphi\,\mathrm{d}\rho=\frac{1}{2\pi}\int_0^{\infty}\frac{J_0(\rho|x|)\rho}{P(\rho^2,p)}\,\mathrm{d}\rho, \tag{6}$$

where $J_0(z)$ is the zeroth-order Bessel function of the first kind.

To evaluate the integral in (6), it is necessary to consider several cases. We start with the case when all four roots $s_i = -\chi_i^2,\ i = 1, 2, 3, 4,$ are distinct. We set

$$a_0 = h^4 \mu^2 (\lambda + 2\mu)$$

and split $1/P(s, p)$ into partial fractions, as

$$\frac{1}{P(s, p)} = \frac{1}{a_0} \sum_{j=1}^{4} c_j (s - s_j)^{-1}.$$

A simple calculation shows that the coefficients

$$c_j = \prod_{\substack{k=1\\ k\neq j}}^{4} (s_j - s_k)^{-1}, \quad j = 1, 2, 3, 4,$$

satisfy the system of linear equations

$$\sum_{k=1}^{4} c_k s_k^m = \delta_{m3}, \quad m = 0, 1, 2, 3, \tag{7}$$

where $\delta_{ij}$ is the Kronecker delta. Hence, since $\mathfrak{Re}\,\chi_j > 0$,

$$\hat{\Psi}(x, p) = \frac{1}{2\pi a_0} \sum_{j=1}^{4} c_j \int_0^\infty \frac{J_0(\rho|x|)\rho}{\rho^2 + \chi_j^2}\, \mathrm{d}\rho = \frac{1}{2\pi a_0} \sum_{j=1}^{4} c_j K_0(\chi_j |x|);$$

here $K_m(z)$ is the modified Bessel function of order $m$, given by [10, Equation 9.6.11]

$$\begin{aligned} K_m(z) = (-1)^{m+1} I_m(z) \log\frac{z}{2} + \frac{1}{2}\sum_{k=0}^{m-1} (-1)^k \frac{(m-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-m} \\ + \frac{(-1)^m}{2} \sum_{k=0}^{\infty} \frac{\psi(k+m+1) + \psi(k+1)}{(m+k)!k!}\left(\frac{z}{2}\right)^{m+2k}, \end{aligned}$$

where

$$I_m(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+m}}{k!\,\Gamma(k+m+1)}, \quad \psi(n+1) = -c + \sum_{k=1}^{n} \frac{1}{k}, \quad \psi(1) = -c,$$

$$c = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Making use of the asymptotic behavior of the function $K_0(z)$ in the neighborhood of $z = 0$, we find that

$$\hat{\Psi}(x, p) = -\frac{1}{2\pi a_0} \sum_{j=1}^{4} c_j \log\frac{\chi_j |x|}{2} \left\{ 1 + \frac{\chi_j^2 |x|^2}{4} + \frac{\chi_j^4 |x|^4}{2^4 (2!)^2} + \frac{\chi_j^6 |x|^6}{2^6 (3!)^2} + \mathrm{O}(|x|^8) \right\} + \hat{\Psi}_1(x, p),$$

where $\hat{\Psi}_1(x, p)$ is infinitely differentiable. Since the coefficients $c_j,\ j = 1, 2, 3, 4,$ satisfy (7), it follows that

$$\hat{\Psi}(x, p) = \frac{1}{2\pi a_0 2^6 (3!)^2} |x|^6 \log|x| + \mathrm{O}(|x|^8 \log|x|) + \hat{\Psi}_0(x, p) \tag{8}$$

with an infinitely differentiable function $\hat{\Psi}_0(x, p)$.

We now consider the case where $s_1 = s_2$ and $s_1, s_3$, and $s_4$ are distinct. Then

$$\frac{1}{P(s,p)} = \frac{1}{a_0}\left\{\sum_{j=1,3,4} \frac{c_j}{s-s_j} + \frac{c_2}{(s-s_1)^2}\right\}$$

and

$$c_1 = \frac{s_3+s_4-2s_1}{(s_1-s_3)^2(s_1-s_4)^2}, \quad c_2 = \frac{1}{(s_1-s_3)(s_1-s_4)},$$

$$c_3 = \frac{1}{(s_1-s_3)^2(s_3-s_4)}, \quad c_4 = \frac{1}{(s_1-s_4)^2(s_4-s_3)},$$

$$\hat{\Psi}(x,p) = \frac{1}{2\pi a_0}\left\{\sum_{j=1,3,4} c_j K_0(\chi_j|x|) + c_2 \frac{|x|}{2\chi_1} K_1(\chi_1|x|)\right\}.$$

A straightforward calculation based on the asymptotic behavior of $K_0(z)$ and $K_1(z)$ shows that the asymptotic behavior of $\hat{\Psi}(x,p)$ is again given by (8).

If $s_1 = s_2$ and $s_3 = s_4$ with distinct $s_1$ and $s_3$, then

$$\frac{1}{P(s,p)} = \frac{1}{a_0}\left\{\sum_{j=1}^{2} \frac{c_j}{(s-s_1)^j} + \sum_{j=3}^{4} \frac{c_j}{(s-s_3)^{j-2}}\right\},$$

and

$$c_1 = \frac{2}{(s_3-s_1)^3}, \quad c_2 = \frac{1}{(s_3-s_1)^2}, \quad c_3 = \frac{2}{(s_1-s_3)^3}, \quad c_4 = \frac{1}{(s_1-s_3)^2},$$

$$\hat{\Psi}(x,p) = \frac{1}{2\pi a_0}\left\{\sum_{j=1,3} c_j K_0(\chi_j|x|) + \frac{|x|}{2} \sum_{j=2,4} \frac{c_j}{\chi_{j-1}} K_1(\chi_{j-1}|x|)\right\}.$$

From this it follows that the asymptotic behavior of $\hat{\Psi}(x,p)$ is once more given by (8).

If $s_1 = s_2 = s_3$ and $s_1$ and $s_4$ are distinct, then

$$\frac{1}{P(s,p)} = \frac{1}{a_0}\left\{\sum_{j=1}^{3} \frac{c_j}{(s-s_1)^j} + \frac{c_4}{s-s_4}\right\},$$

and

$$c_1 = \frac{1}{(s_1-s_4)^3}, \quad c_2 = -\frac{1}{(s_1-s_4)^2},$$

$$c_3 = \frac{1}{s_1-s_4}, \quad c_4 = -\frac{1}{(s_1-s_4)^3},$$

$$\hat{\Psi}(x,p) = \frac{1}{2\pi a_0}\left\{\sum_{j=1}^{2} c_j \left(\frac{|x|}{2\chi_1}\right)^{j-1} K_{j-1}(\chi_1|x|) + \frac{c_3}{2}\left(\frac{|x|}{2\chi_1}\right)^2 K_2(\chi_1|x|) + c_4 K_0(\chi_4|x|)\right\},$$

which immediately leads to (8).

Finally, if $s_1 = s_2 = s_3 = s_4$, then

$$\frac{1}{P(s,p)} = \frac{1}{a_0}\frac{1}{(s-s_1)^4},$$

and

$$\hat{\Psi}(x,p) = \frac{1}{2\pi a_0}\frac{1}{6}\left(\frac{|x|}{2\chi_1}\right)^3 K_3(\chi_1|x|).$$

Again, direct computation shows that $\hat{\Psi}(x,p)$ has the representation (8).

In conclusion, we write the entries of $\hat{D}(x,p)$ as

$$\begin{aligned}
\hat{D}_{11}(x,p) &= \big\{(\varkappa^{-1}p-\Delta)\big[(\rho p^2-\mu\Delta)\big(\rho h^2p^2-h^2\mu\Delta+\mu-h^2(\lambda+\mu)\partial_2^2\big)+\mu^2\partial_2^2\big]-\\
&\quad -\eta h^2\gamma p\partial_2^2(\rho p^2-\mu\Delta)\big\}\hat{\Psi}(x,p),\\
\hat{D}_{22}(x,p) &= \big\{(\varkappa^{-1}p-\Delta)\big[(\rho p^2-\mu\Delta)\big(\rho h^2p^2-h^2\mu\Delta+\mu-h^2(\lambda+\mu)\partial_1^2\big)+\mu^2\partial_1^2\big]-\\
&\quad -\eta h^2\gamma p\partial_1^2(\rho p^2-\mu\Delta)\big\}\hat{\Psi}(x,p),\\
\hat{D}_{33}(x,p) &= \big\{(\varkappa^{-1}p-\Delta)\big[\big(\rho h^2p^2-h^2\mu\Delta+\mu-h^2(\lambda+\mu)\partial_1^2\big)\times\\
&\qquad\times\big(\rho h^2p^2-h^2\mu\Delta+\mu-h^2(\lambda+\mu)\partial_2^2\big)-h^4(\lambda+\mu)^2\partial_1^2\partial_2^2\big]-\\
&\quad -\eta h^2\gamma p\,\Delta(\rho h^2p^2-h^2\mu\Delta+\mu)\big\}\hat{\Psi}(x,p),\\
\hat{D}_{44}(x,p) &= (\rho h^2p^2-h^2\mu\Delta+\mu)\big[(\rho p^2-\mu\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\\
&\qquad -h^2(\lambda+\mu)\Delta(\rho p^2-\mu\Delta)+\mu^2\Delta\big]\hat{\Psi}(x,p),\\
\hat{D}_{12}(x,p) &= \hat{D}_{21}(x,p)=\big\{(\varkappa^{-1}p-\Delta)\big[h^2(\lambda+\mu)(\rho p^2-\mu\Delta)-\mu^2\big]\\
&\qquad +\eta h^2\gamma p(\rho p^2-\mu\Delta)\big\}\partial_1\partial_2\hat{\Psi}(x,p),\\
\hat{D}_{13}(x,p) &= -\hat{D}_{31}(x,p)=-\mu(\varkappa^{-1}p-\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_1\hat{\Psi}(x,p),\\
\hat{D}_{14}(x,p) &= -h^2\gamma(\rho p^2-\mu\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_1\hat{\Psi}(x,p),\\
\hat{D}_{41}(x,p) &= -\eta p(\rho p^2-\mu\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_1\hat{\Psi}(x,p),\\
\hat{D}_{23}(x,p) &= -\hat{D}_{32}(x,p)=-\mu(\varkappa^{-1}p-\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_2\hat{\Psi}(x,p),\\
\hat{D}_{24}(x,p) &= -h^2\gamma(\rho p^2-\mu\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_2\hat{\Psi}(x,p),\\
\hat{D}_{42}(x,p) &= -p\eta(\rho p^2-\mu\Delta)(\rho h^2p^2-h^2\mu\Delta+\mu)\partial_2\hat{\Psi}(x,p),\\
\hat{D}_{34}(x,p) &= -h^2\gamma\mu(\rho h^2p^2-h^2\mu\Delta+\mu)\Delta\hat{\Psi}(x,p),\\
\hat{D}_{43}(x,p) &= \mu\eta p(\rho h^2p^2-h^2\mu\Delta+\mu)\Delta\hat{\Psi}(x,p).
\end{aligned}$$

Clearly, these formulas together with (8) make it possible to determine the asymptotic behavior of all entries $\hat{D}_{ij}(x,p)$, $i,j=1,2,3,4$, in the neighborhood of $x=0$.

Figures 1 and 2 offer a graphical comparison of the singularities at the origin of $\hat{D}_{11}(x,p)$ and $-\log|x|$ for

$$\begin{aligned}
&\lambda=\mu=\rho=\gamma=\eta=\varkappa=h=p=1,\\
&\quad x,y\in[-1,1]\times[-1,1]\setminus(0,0).
\end{aligned}$$

## 5. Conclusions

In this paper, we have completed the first stage in the construction of boundary-element methods for the bending motion of thermoelastic plates with transverse shear deformation, which consists in studying the singularity at the origin of the Laplace transform $\hat{D}(x,p)$ of a matrix $D(x,t)$ of fundamental solutions for the governing system of the model. As Figures 1 and 2 suggest, we expect the formal asymptotic expansions of the entries of $\hat{D}(x,p)$ to show that, in fact, $\hat{D}(x,p)=O(\log|x|)$ as $|x|\to 0$.

Once this is confirmed analytically, we can then proceed to the second stage, where we use $\hat{D}(x,p)$ to construct single-layer and double-layer potentials for the transformed system.

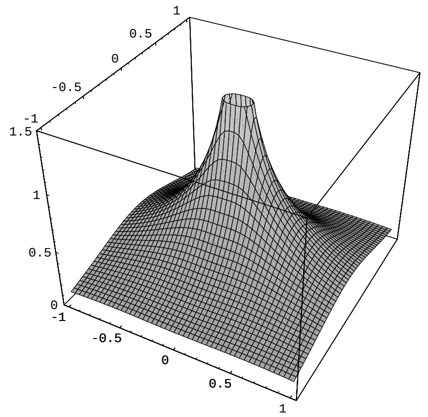


*Figure 1.* Graph of $\hat{D}_{11}(x, p)$.

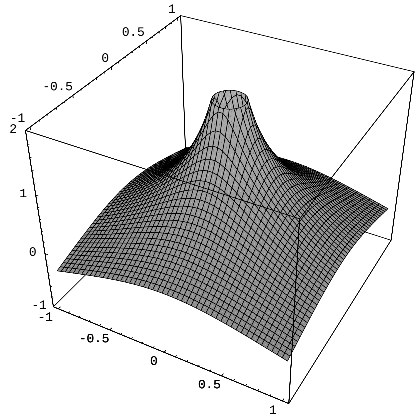


*Figure 2.* Graph of $-\log|x|$.

The Laplace transformation reduces initial-boundary-value problems for the original time-dependent system to boundary-value problems for the transformed one, which, in turn, are then reduced to boundary-integral equations for the densities of the layer potentials. By using functional-analysis techniques to study the mapping properties of the boundary operators associated with the integral equations, we intend to prove that these equations have unique solutions in appropriate spaces of distributions, and that the solutions are holomorphic functions with respect to the transformation parameter $p$. Further analysis will then be expected to show that the inverse transforms of the potentials with these densities are solutions of the original initial-boundary-value problems.

In the third stage of the investigation, boundary-element (Galerkin, collocation) methods will be rigorously designed to compute approximate solutions of the relevant integral equations and, thus, approximate solutions to the given initial-boundary-value problems.

## Appendix A

We prove that $\mathfrak{Re}\chi_j > 0,\ j = 1, 2, 3, 4$.

First, we show that $\chi_j \neq 0,\ j = 1, 2, 3, 4$. True, if $\chi_j = s_j = 0$ for some $j$, then

$$P(s_j, p) = \rho\varkappa^{-1} p^3 (\rho h^2 p^2 + \mu)^2 = 0$$

and $\mathfrak{Re}p = 0$, which contradicts our assumption that $\mathfrak{Re}p > 0$.

Suppose that $s_j = s = -\chi^2$, $\mathfrak{Re}\chi = 0$, for some $p$. Then for $s > 0$, the system of equations

$$\begin{pmatrix} \rho h^2 p^2 + h^2(\lambda + 2\mu)s + \mu & 0 & -\mathrm{i}\mu\sqrt{s} & -\mathrm{i}h^2\gamma\sqrt{s} \\ 0 & \rho h^2 p^2 + h^2\mu s + \mu & 0 & 0 \\ \mathrm{i}\mu\sqrt{s} & 0 & \rho p^2 + \mu s & 0 \\ -\mathrm{i}\eta p\sqrt{s} & 0 & 0 & \varkappa^{-1}p + s \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = 0 \quad \text{(A1)}$$

has a nontrivial solution $g = (g_1, g_2, g_3, g_4)^{\mathrm{T}} \in \mathbb{C}^4$. To prove this, it suffices to recall that $\det\Theta(\xi, p)$ depends only on $|\xi|^2 = s$, and to set $\xi_1 = |\xi|^2 = s$ and $\xi_2 = 0$ in the system of equations $\Theta(\xi, p)g = 0$. Then (A1) is written in the form

$$\begin{pmatrix} (\rho h^2 p^2 + h^2(\lambda + 2\mu)s + \mu)g_1 - \mathrm{i}\mu\sqrt{s}\, g_3 - \mathrm{i}h^2\gamma\sqrt{s}\, g_4 \\ (\rho h^2 p^2 + h^2\mu s + \mu)g_2 \\ \mathrm{i}\mu\sqrt{s}\, g_1 + (\rho p^2 + \mu s)g_3 \\ -\mathrm{i}\eta p\sqrt{s}\, g_1 + (\varkappa^{-1}p + s)g_4 \end{pmatrix} = 0. \tag{A2}$$

From (A2) it follows that

$$0=\left(\bar{g}_1,\bar{g}_2,\bar{g}_3,-\frac{h^2\gamma}{\eta p}\,\bar{g}_4\right)\begin{pmatrix}(\rho h^2p^2+h^2(\lambda+2\mu)s+\mu)g_1-\mathrm{i}\mu\sqrt{s}\,g_3-\mathrm{i}h^2\gamma\sqrt{s}\,g_4\\(\rho h^2p^2+h^2\mu s+\mu)g_2\\\mathrm{i}\mu\sqrt{s}\,g_1+(\rho p^2+\mu s)g_3\\-\mathrm{i}\eta p\sqrt{s}\,g_1+(\varkappa^{-1}p+s)g_4\end{pmatrix}$$

$$=\big(\rho h^2p^2+h^2(\lambda+2\mu)s+\mu\big)|g_1|^2+(\rho h^2p^2+h^2\mu s+\mu)|g_2|^2+(\rho p^2+\mu s)|g_3|^2$$

$$-\frac{h^2\gamma}{\eta p}(\varkappa^{-1}p+s)|g_4|^2-\mathrm{i}\mu\sqrt{s}\,\bar{g}_1g_3+i\mu\sqrt{s}\,g_1\bar{g}_3-ih^2\gamma\sqrt{s}\,\bar{g}_1g_4+ih^2\gamma\sqrt{s}\,g_1\bar{g}_4;$$

therefore,

$$\rho h^2p^2(|g_1|^2+|g_2|^2)+\rho p^2|g_3|^2-\frac{h^2\gamma s}{\eta p}|g_4|^2=-(h^2(\lambda+2\mu)s+\mu)|g_1|^2-\mu(h^2s+1)|g_2|^2$$

$$-\mu s|g_3|^2+\frac{\varkappa^{-1}h^2\gamma}{\eta}|g_4|^2-2\mathfrak{Re}\big(\mathrm{i}\mu\sqrt{s}\,g_1\bar{g}_3+\mathrm{i}h^2\gamma\sqrt{s}\,g_1\bar{g}_4\big)\in\mathbb{R}. \tag{A3}$$

Let $p=\sigma+\mathrm{i}\tau$, $\sigma>0$. The imaginary part of the left-hand side in (A3) is zero; hence,

$$2\rho h^2\sigma\tau(|g_1|^2+|g_2|^2)+2\rho\sigma\tau|g_3|^2+\frac{h^2\gamma s\tau}{\eta|p|^2}|g_4|^2=0. \tag{A4}$$

From (A4) it follows that $\tau=0$ and that $p=\sigma>0$.

Next, we analyze system (A2). Clearly, the second equation implies that $g_2=0$. If we add the third equation multiplied by $\mathrm{i}\mu\sqrt{s}/(\rho p^2+\mu s)$ and the fourth multiplied by $\mathrm{i}h^2\gamma\sqrt{s}/(\varkappa^{-1}p+s)$ to the first equation, we deduce that $g_1=0$. The second and third equations then yield $g_3=g_4=0$. This contradiction completes the proof.

We remark that one of the roots of $P(s,p)$ is

$$s=-\frac{1}{h^2\mu}(\rho h^2p^2+\mu);$$

the other three roots may be computed, for example, by means of Cardan's formulas.

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